Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-12-03

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to warm-up questions

Question 1. This is false. A normal vector to the surface $z = f(x, y)$ at the point where $x = a$ and $y = b$ is given by $\langle f_x(a, b), f_y(a, b), -1\rangle$.

The gradient $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$ gives a normal vector to the *level set* $k = f(x, y)$ in the plane $\mathbb{R}^2$ though, where k is the constant $f(a, b)$.

Question 2. This is true. You can see it by differentiating the equation $\mathbf{r} \cdot \mathbf{r} = 1$ with respect to t.

Answers to computations

I will mostly just provide outlines of the solutions. If you want more details or catch any mistakes, just let me know.

Problem 1. The angle between two planes is the angle between their normal vectors, but note that you should give an angle between 0 and $\pi/2$ for your final answer. So if the angle you computed between the normal vectors is obtuse, take π minus that angle instead.

Problem 2. The point $(-16, 4)$ corresponds to $t = 2$. We have $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 2$, which means that our tangent line is vertical: $x = -16$ is its Cartesian equation.

Problem 3. At $(0, 0)$, we have $f(0, 0) = 0$, $f_x(0, 0) = 1$, $f_y(0, 0) = 1$. So the linear approximation is

$$L(x, y) = 0 + 1(x - 0) + 1(y - 0) = x + y.$$

Problem 4. Let's try and find a potential function f in the standard systematic way:

$$f_x(x, y, z) = \frac{1}{x^2} + y$$

so $f(x, y, z) = -\frac{1}{x} + xy + g(y, z)$ for some function g of y and z. Differentiating this with respect to y and comparing to $\mathbf{F}$ yields

$$f_y(x, y, z) = x + g_y(y, z) = x + \frac{1}{y} + z^2$$

hence $g_y(y, z) = \frac{1}{y} + z^2$, meaning $g(y, z) = \ln|y| + yz^2 + h(z)$ where h is a function depending only on z. Finally, differentiating $f(x, y, z) = -\frac{1}{x} + xy + \ln|y| + yz^2 + h(z)$ with respect to z and comparing to $\mathbf{F}$ gives

$$f_z(x, y, z) = 2yz + h'(z) = 2yz$$

so we need $h'(z) = 0$. So let's just take $h(z) = 0$.

We didn't run into any impossible situations when doing this, so our vector field $\mathbf{F}$ is conservative and this function $f(x, y, z) = -\frac{1}{x} + xy + \ln|y| + yz^2$ is a potential function.

Problem 5. Find the points where $\nabla f(x, y) = \langle 0, 0\rangle$ in the interior region $x^4 + y^4 < 1$, and then use Lagrange multipliers to deal with the boundary $x^4 + y^4 = 1$. You should find $(0, 0)$ in the inside, and the candidates along the boundary are $(-1, 0), (1, 0), (0, -1), (0, 1), (-2^{-1/4}, -2^{-1/4}), (2^{-1/4}, 2^{-1/4})$. The max value is $2^{1/4}$, attained at the last point.

Problem 6. If you remember the polar arc length formula, just use that. If not, then no problem—any polar curve can just be converted to a parametric curve; for instance this one is

$$x = (\theta^2 - 1)\cos\theta,\ \ y = (\theta^2 - 1)\sin\theta.$$

Then you can evaluate $\int_C \mathrm{d}s$ in the usual way.

Problem 7. Same deal as the previous problem. In fact I strongly suggest you don't bother remembering the formula for $\mathrm{d}y/\mathrm{d}x$ in polar; just convert your polar curve to a parametric curve and use the formula for $\mathrm{d}y/\mathrm{d}x$ for parametric curves.

Problem 8. You'll need to do this by direct parametrization rather than any fancy theorems.

Problem 9. The integrand is a conservative vector field; it's the gradient of the product fg. Check this! Hence the integral is zero. This argument works regardless of whether f, g are defined on all of $\mathbb{R}^3$.

If you computed $\nabla \times (f\nabla g + g\nabla f) = \mathbf{0}$, that would show the the integrand is conservative on any simply connected region, so it would work for the first half of the problem but not the second.

Problem 10. Direct computation shows that it's a critical point. If you try to use the second derivative test, you'll get $D = 0$, so it's inconclusive. But $f(0,0) = 1$ and clearly $f(x, y) \le 1$ for all (x, y), so $(0,0)$ is a local max (indeed, it is a global max).

Problem 11. Rewrite the plane as $z = \frac{1}{3}(1 - x - 2y)$ and parametrize using x, y. The region of integration in the xy-plane will be the disk D given by $x^2 + y^2 \le 3$. At that point you can convert to polar (or use geometry, noting the area of the disk is 3π).

Problem 12. We are told

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$$

which expands to

$$\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0.$$

But $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, so this formula just says

$$|\mathbf{a}|^2 = |\mathbf{b}|^2.$$

Since the lengths are obviously nonnegative, this means we must have $|\mathbf{a}| = |\mathbf{b}|$.

Problem 13. Switch the order of integration.

Problem 14. It's zero, because the first term doesn't involve z, the second doesn't involve y, and the third doesn't involve x.

Problem 15. Let E be the solid ellipsoid. If we apply the change of variables $x = au$, $y = bv$, $z = cw$, then

$$\iiint_E 1 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iiint_{u^2+v^2+w^2 \le 1} abc \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w$$

because abc is the absolute value of the Jacobian determinant. So the final answer is just $\frac{4}{3}\pi abc$.

Problem 16. Let's do $\mathrm{d}y$ first because it only shows up in two of the bounding planes, while x, z show up in all 4. We have

$$x + z - 1 \le y \le 1 - x - z$$

as the bounds. The region in the xz-plane is bounded by the lines $x = 0$, $z = 0$, and $x + z = 1$. This triangular region can be set up easily either as $\mathrm{d}x \, \mathrm{d}z$ or $\mathrm{d}z \, \mathrm{d}x$ so I'll just do the former. Altogether we get

$$\int_0^1 \int_0^{1-z} \int_{x+z-1}^{1-x-z} 1 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z.$$

Problem 17. Use the chain rule. One approach is to use $x = r\cos\theta$, $y = r\sin\theta$, giving you the equations

$$\begin{aligned} 4 &= f_x(6,8)(3/5) + f_y(6,8)(4/5) \\ -2 &= f_x(6,8)(-8) + f_y(6,8)(6) \end{aligned}$$

which you can then solve for $f_x(6,8)$ and $f_y(6,8)$.

Alternatively note that we have $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$, so applying the chain rule gives

$$\begin{aligned} f_x(6,8) &= (-2)(-8/100) + (4)(3/5) \\ f_y(6,8) &= (-2)(6/100) + (4)(4/5). \end{aligned}$$

The final answer is $f_x(6,8) = 64/25$ and $f_y(6,8) = 77/25$.

Problem 18. Let D denote the region. The center of mass has coordinates $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\iint_D x\sigma(x,y) \, \mathrm{d}x \, \mathrm{d}y}{\iint_D \sigma(x,y) \, \mathrm{d}x \, \mathrm{d}y}, \bar{y} = \frac{\iint_D y\sigma(x,y) \, \mathrm{d}x \, \mathrm{d}y}{\iint_D \sigma(x,y) \, \mathrm{d}x \, \mathrm{d}y}.$$

By symmetry one can argue that $\iint_D y|y| \, \mathrm{d}x \, \mathrm{d}y$ is zero. So it just remains to compute $\bar{x}$. Again using symmetry we have

$$\bar{x} = \frac{2\iint_{D_+} xy \, \mathrm{d}x \, \mathrm{d}y}{2\iint_{D_+} y \, \mathrm{d}x \, \mathrm{d}y}$$

where D_+ denotes the half of the region above the x-axis.

Problem 19. As suggested, consider the equation $x^7 - ax^6 + bx - 2 = 0$. Call the left hand side $F(a, b, x)$, where x depends on a, b. Then

$$\frac{\partial x}{\partial a} = -\frac{F_a}{F_x}$$
$$\frac{\partial x}{\partial b} = -\frac{F_b}{F_x}.$$

Now just evaluate $F_a(1,2,1), F_b(1,2,1), F_x(1,2,1)$:

$$F_a(1,2,1) = -(1)^6 = -1$$
$$F_b(1,2,1) = (1) = 1$$
$$F_x(1,2,1) = 7(1)^6 - 6(1)(1)^5 + (2) = 3$$

so by linear approximation we have that when $a = 1.03$ and $b = 2.06$, a solution for x is approximately

$$1 + \frac{1}{3}(0.03) - \frac{1}{3}(0.06) = 0.99.$$

Problem 20. The integrand and the region strongly suggests the change of variables $u = y - 2x, v = y + 3x$. The corresponding region in the uv-plane is then bounded by the lines

$$u = 1, \ u = 4, \ v = 1, \ v = 4.$$

Also we have

$$\mathrm{d}u\,\mathrm{d}v = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y = \left|\det\begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}\right| \mathrm{d}x\,\mathrm{d}y = 5\,\mathrm{d}x\,\mathrm{d}y$$

which means $\mathrm{d}x\,\mathrm{d}y = \frac{1}{5}\,\mathrm{d}u\,\mathrm{d}v$. Hence

$$\iint_D \frac{y-2x}{y+3x}\,\mathrm{d}x\,\mathrm{d}y = \int_1^4 \int_1^4 \frac{u}{5v}\,\mathrm{d}u\,\mathrm{d}v.$$

(In this problem we were able to avoid having to solve x, y in terms of u, v, but in general you may have to do that.)

Problem 21. It would be annoying to figure out exactly what this circle is parametrically, but if we let S denote the region enclosed by the circle in the plane $x + 2y + z = 4$, we know that the area of S is just π, i.e.

$$\iint_S 1\,\mathrm{d}S = \pi.$$

Since we know this, it would be wise to try and convert the line integral to a surface integral via Stokes':

$$\int_C \langle y, 2x, (2x-y)\rangle \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \langle y, 2x, (2x-y)\rangle) \cdot \mathrm{d}\mathbf{S} = \iint_S \langle -1, -2, 1\rangle \cdot \mathrm{d}\mathbf{S}.$$

For this to make sense, we need to orient S correctly. Since C is oriented clockwise when viewed from the origin, the RHR tells us that we must orient S with a normal vector pointing *away* from the origin for Stokes' Theorem to hold as we've written it.

Normally to directly compute a surface integral like this, you want to parametrize. But as mentioned before, the region enclosed by C is a bit annoying to write down bounds for. Since we already know $\iint_S 1\,\mathrm{d}S = \pi$, let's just use $\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S$ to rewrite our flux integral. A unit normal for S is $\langle 1,2,1\rangle/\sqrt{6}$, and this points away from the origin as we want. To check this, you could take a point on S, e.g. $(0,2,0)$, and then note that $\langle 0,2,0\rangle \cdot \langle 1,2,1\rangle/\sqrt{6} > 0$ (why does this imply that $\mathbf{n}$ points away from the origin?). So

$$\iint_S \langle -1,-2,1\rangle \cdot \frac{\langle 1,2,1\rangle}{\sqrt{6}}\,\mathrm{d}S = -\frac{4}{\sqrt{6}} \iint_S \mathrm{d}S = -\frac{4\pi}{\sqrt{6}}.$$

Problem 22.

(a) The Lagrange multipliers system is

$$u = \frac{1}{3}\lambda x^{-2/3} y^{2/3}$$
$$v = \frac{2}{3}\lambda x^{1/3} y^{-1/3}$$
$$x^{1/3} y^{2/3} = 1.$$

(Note that x, y must be nonzero because of the constraint equation, so these negative exponents are fine.) We want to get rid of λ, so multiply the first equation by $2x$ and the second equation by y to get

$$2ux = \frac{2}{3}\lambda x^{1/3}y^{2/3} = vy.$$

Then just solve the system

$$\begin{aligned} 2ux &= vy \\ x^{1/3}y^{2/3} &= 1 \end{aligned}$$

for x, y in terms of u, v.

(b) If you compute the Jacobian determinant after solving (a), you'll find that the answer is zero. This is because (x, y) is constrained to a curve, namely $x^{1/3}y^{2/3} = 1$. So the transformation from the uv-plane to the xy-plane maps to a lower-dimensional region, and must have zero Jacobian determinant (because it multiplies areas by zero).

Problem 23. Let $F(x, y, u) = u - h(x - yu)$, so that our functional equation is $F(x, y, u) = 0$. Implicit differentiation yields

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{F_x}{F_u} = -\frac{-h'(x-yu)}{1-h'(x-yu)(-y)} \\ \frac{\partial u}{\partial y} &= -\frac{F_y}{F_u} = -\frac{-h'(x-yu)(-u)}{1-h'(x-yu)(-y)} \end{aligned}$$

and the desired identity easily follows from this. (So it didn't really matter what $h(t)$ actually was.)

Problem 24. Let D be the portion of S of interest. We can parametrize D using cylindrical coordinates, noting that the sphere equation gives $r = \sqrt{R^2 - z^2}$:

$$\mathbf{r}(\theta, z) = \langle \sqrt{R^2 - z^2}\cos\theta, \sqrt{R^2 - z^2}\sin\theta, z\rangle$$

where $0 \le \theta \le 2\pi$ and $a \le z \le a + h$.

So the surface area of D is

$$\int_0^{2\pi}\int_a^{a+h} |\mathbf{r}_\theta \times \mathbf{r}_z|\, \mathrm{d}\theta\, \mathrm{d}z.$$

Problem 25. Parametrize S as $\mathbf{r}(\theta, \phi) = \langle f(\theta)\sin\phi\cos\theta, f(\theta)\sin\phi\sin\theta, f(\theta)\cos\phi\rangle$ so the surface area is

$$\iint_D |\mathbf{r}_\theta \times \mathbf{r}_\phi|\, \mathrm{d}\theta\, \mathrm{d}\phi.$$

The rest of the problem is computing $|\mathbf{r}_\theta \times \mathbf{r}_\phi|$, which is very tedious.

If you are wondering if there's a better way of doing this, there is, but it requires some knowledge of linear algebra.

Problem 26. Let E be the 3D region $x^2 + y^2 + z^2 \le 1$, $z \ge 0$. The boundary ∂E consists of the hemisphere S oriented upwards (which is the surface we care about) together with the disk $z = 0$, $x^2 + y^2 \le 1$ oriented downwards. Call this disk D. So the Divergence Theorem states

$$\begin{aligned} \iiint_E (\nabla \cdot \langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z\rangle)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z &= \iint_S \langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z\rangle \cdot \mathrm{d}\mathbf{S} \\ &\quad + \iint_D \langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z\rangle \cdot \mathrm{d}\mathbf{S}. \end{aligned}$$

We have $\nabla \cdot \langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z\rangle = 1$, so the left side is just the volume of E, and you can compute that by e.g. using a triple integral

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta = \frac{2}{3}\pi.$$

D can be parametrized just as $\mathbf{r} = \langle x, y, 0\rangle$. Note that $\mathbf{r}_x \times \mathbf{r}_y = \langle 0, 0, 1\rangle$ but we need to take its negative since D is to be oriented downwards.

$$\begin{aligned} \iint_D \langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z\rangle \cdot \mathrm{d}\mathbf{S} &= \iint_{x^2+y^2\le 1} \langle \cdots, \cdots, \arctan(x) - 0\rangle \cdot \langle 0, 0, -1\rangle\, \mathrm{d}x\, \mathrm{d}y \\ &= \iint_{x^2+y^2\le 1} (-\arctan(x))\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (-\arctan(x))\, \mathrm{d}x\, \mathrm{d}y = \int_{-1}^{1} 0\, \mathrm{d}y = 0 \end{aligned}$$

because $\arctan(x)$ is an odd function.

So in the end,

$$\iint_S \langle 3 + e^{yz}, 2y + \sin(xz), \arctan(x) - z \rangle \cdot d\mathbf{S} = \frac{2\pi}{3}.$$

Problem 27. The given region is really inconvenient to set up in any of the coordinate systems that we are familiar with. We have two spheres, both of radius 2, one centered at $(2, 0, 0)$ and the other centered at $(0, 0, 2)$.

But geometrically, all we have is **two spheres of radius 2, whose centers are** $2\sqrt{2}$ **units apart**. The idea is throw away the original coordinate description of this problem, and set it up so that instead we have one sphere centered at $(0, 0, 0)$, and the other centered at $(0, 0, 2\sqrt{2})$. In other words, It's more convenient to deal with the equivalent volume

$$x^2 + y^2 + z^2 \le 4$$
$$x^2 + y^2 + (z - 2\sqrt{2})^2 \le 4.$$

Note that these new spheres intersect in a circle contained in the plane $z = \sqrt{2}$. The region is symmetric across this plane, so our final answer will be 2 times the volume of

$$x^2 + y^2 + z^2 \le 4$$
$$z \ge \sqrt{2}.$$

In this form, this is a problem that you should be comfortable doing, e.g. with cylindrical coordinates.